One-arm Exponent of Critical Level-set for Metric Graph Gaussian Free Field in High Dimensions

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The 18th Workshop on Markov Process and Related Topics

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Definition to the metric graph GFF

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- Discrete Gaussian free field (DGFF) on $\mathbb{Z}^d$ ($d \geq 3$): a mean-zero Gaussian field $\{\phi_x\}_{x \in \mathbb{Z}^d}$ with

$$\mathbb{E}\left(\phi_{x_1}\phi_{x_2}\right) = G(x_1, x_2),\ \forall x_1, x_2 \in \mathbb{Z}^d$$

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  where $G(\cdot, \cdot)$ is the Green's function on $\mathbb{Z}^d$.
- Metric graph $\widetilde{\mathbb{Z}}^d$: For any adjacent points $x, y \in \mathbb{Z}^d$, let $I_{\{x,y\}}$ be a compact interval of length d with two endpoints identical to x and y respectively. Then define

$$\widetilde{\mathbb{Z}}^d := \bigcup_{\text{adjacent } x,y \in \mathbb{Z}^d} I_{\{x,y\}}.$$

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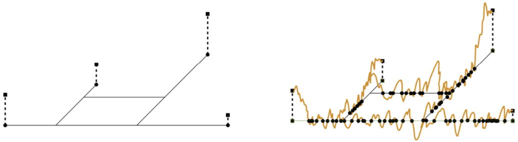
2. In each interval $I_{\{x,y\}}$, $\{\widetilde{\phi}_v\}_{v \in I_{\{x,y\}}}$ is given by an independent Brownian bridge of length d with variance 2 at time 1, conditioned on $\widetilde{\phi}_x = \phi_x$ and $\widetilde{\phi}_y = \phi_y$.



Illustrations from Lupu-Werner'18.

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- (1) implies that the (critical) one-arm probability

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Questions: How fast does it decay?

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Remark 2: The bounds in (3) are getting rougher as d increases.

Main result

Theorem (Cai-D.'2023)

For $d > 6$, there exist constants $C_1(d), C_2(d) > 0$ such that

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(5) was proved only for $d \ge 11$ by Hara-Slade'90, Fitzner-van der Hofstad'17.

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Lupu's coupling and two-point function

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- Tupu'16: an explicit formula of the two-point function

$$\mathbb{P}\Big[x \xleftrightarrow{\cup \widetilde{\mathcal{L}}_{1/2}} y\Big] = \frac{2}{\pi} \arcsin\left(\frac{G(x,y)}{\sqrt{G(x,x)G(y,y)}}\right) \asymp |x-y|^{2-d}. \quad (6)$$

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- Lupu'16: a coupling between the GFF $\{\widetilde{\phi}_v\}_{v\in\widetilde{\mathbb{Z}}^d}$ and the loop soup $\widetilde{\mathcal{L}}_{1/2}$ on $\widetilde{\mathbb{Z}}^d$ such that a.s.

$$\text{the sign clusters of } \{\widetilde{\phi}_v\}_{v\in\widetilde{\mathbb{Z}}^d} \;=\; \text{the loop clusters of } \widetilde{\mathcal{L}}_{1/2}. \quad (7)$$

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- Let $\cup\widetilde{\mathcal{L}}_{1/2}$ be the union of all loops in $\widetilde{\mathcal{L}}_{1/2}$. (7) implies that

$$\theta(N) = \frac{1}{2}\mathbb{P}\Big[\mathbf{0} \xleftrightarrow{\text{sign clusters of } \widetilde{\phi}} \partial B(N)\Big] = \frac{1}{2}\mathbb{P}\Big[\mathbf{0} \xleftrightarrow{\cup\widetilde{\mathcal{L}}_{1/2}} \partial B(N)\Big]. \quad (8)$$

Remark: Our proof is based on the loop soup $\widetilde{\mathcal{L}}_{1/2}$.

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1. We abbreviate "$\xleftrightarrow{\cup\widetilde{\mathcal{L}}_{1/2}}$" as "$\leftrightarrow$". Let $X := \sum_{x\in\partial B(N)} \mathbb{1}_{\mathbf{0}\leftrightarrow x}$.
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2. By the two-point function estimate, $\mathbb{E}(X) \geq cN$.
3. (tree expansion) For any $x, y \in \partial B(N)$, on the event $\{\mathbf{0} \leftrightarrow x, \mathbf{0} \leftrightarrow y\}$, there exist a loop $\widetilde{\ell}$ in $\widetilde{\mathcal{L}}_{1/2}$ and $z_1, z_2, z_3 \in \mathbb{Z}^d$ such that $\widetilde{\ell}$ intersects $B_{z_1}(1)$, $B_{z_2}(1)$ and $B_{z_3}(1)$, and that $\{z_1 \leftrightarrow \mathbf{0}\}$, $\{z_2 \leftrightarrow x\}$, $\{z_3 \leftrightarrow y\}$ happen disjointly.

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4. By the van den Berg-Kesten-Reimer (BKR) inequality,

$$\begin{aligned}\mathbb{E}(X^2) = & \sum_{x,y\in\partial B(N)} \mathbb{P}\left[\mathbf{0} \leftrightarrow x, \mathbf{0} \leftrightarrow y\right] \\ \leq & C \sum_{x,y\in\partial B(N)} \sum_{z_1,z_2,z_3} |z_1 - z_2|^{2-d}|z_2 - z_3|^{2-d}|z_3 - z_1|^{2-d} \\ & \cdot |z_1|^{2-d}|z_2 - x|^{2-d}|z_3 - y|^{2-d} \\ \leq & CN^4. \end{aligned} \tag{9}$$

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- Triangle condition (also for the bond percolation):

$$\sum_{x,y\in\mathbb{Z}^d} \mathbb{P}\left[\mathbf{0} \leftrightarrow x\right] \mathbb{P}\left[\mathbf{0} \leftrightarrow y\right] \mathbb{P}\left[x \leftrightarrow y\right] < \infty \iff d > 6.$$

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- Moreover, we need $d > 6$ for some technical inequalities, which are used repeatedly in our proof (such as (9)). For example,

$$\sum_{z\in\mathbb{Z}^d} |z-x|^{2-d}|z-y|^{2-d} \le C|x-y|^{4-d},\ \forall x,y\in\mathbb{Z}^d \iff d > 6.$$

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- Werner'2021 conjectured that for $d \in \{3,4,5\}$, the large loop cluster is typically formed by gluing macroscopic loops with microscopic loops (the gluing mechanism differs according to the dimension).
- Werner'2021 also presented heuristics on proving that for $d > 6$, the large loop cluster is typically composed of microscopic loops.

Proof of the upper bound 0: Gady-Nachmias framework

- To obtain $\theta(N) \leq CN^{-2}$ by induction, it suffices to prove that for any sufficiently small $\lambda, \epsilon > 0$ and sufficiently large N.

$$\theta((1+\lambda)N) \leq \frac{C_3}{[(1+\lambda)N]^{2+c_1}} + \frac{C_4}{\epsilon^{\frac{1}{2}} N^2} + C_5 \epsilon^{\frac{3}{5}} N^2 \theta(\tfrac{\lambda N}{2}) \theta(N) + (1-c_2)\theta(N).$$

- In fact, $\{\mathbf{0} \leftrightarrow \partial B((1+\lambda)N)\}$ can be divided into four sub-events B_0, B_1, B_2 and B_3, which correspond to the four terms on the RHS above.

Proof of the upper bound 1: deleting large loops

- Take a constant $b \in (\frac{6}{d}, 1)$. Define

$$\mathsf{B}_0 := \left\{\mathbf{0} \leftrightarrow \partial B((1+\lambda)N)\right\} \cap \left\{\mathbf{0} \xleftrightarrow{\le[(1+\lambda)N]^b} \partial B((1+\lambda)N)\right\}^c,$$

where "$\xleftrightarrow{\le[(1+\lambda)N]^b}$" means "connected by $\widetilde{\mathcal{L}}_{1/2} \cdot \mathbb{1}_{\mathrm{diam}(\widetilde{\ell}) \le [(1+\lambda)N]^b}$".

- Following Werner's heuristics, we proved that

$$\mathbb{P}\left[\mathsf{B}_0\right] \le \frac{C_3}{[(1+\lambda)N]^{2+c_1}}.$$

- For any $v \in \widetilde{\mathbb{Z}}^d$, let $\mathbf{C}(v)$ be the cluster of $\cup\widetilde{\mathcal{L}}_{1/2}$ containing x.
- For any $A \subset \widetilde{\mathbb{Z}}^d$, let $|A|$ is the number of lattice points in A.
- We define

$$\mathbf{B}_1 := \left\{ \left|\mathbf{C}(\mathbf{0})\right| \geq \epsilon N^4 \right\}.$$

- Inspired by Barsky-Aizenman'1991, we proved that

$$\mathbb{P}\left[\mathsf{B}_1\right] \leq \frac{C_4}{\epsilon^{\frac{1}{2}} N^2}.$$

Proof of the upper bound 3: a partial cluster as a cut set

- We define a sub-cluster $\widehat{\mathbf{\Psi}}_n$ of $\mathbf{C}(\mathbf{0})$. Roughly speaking (but not accurate), $\widehat{\mathbf{\Psi}}_n$ is the cluster containing $\mathbf{0}$ and composed of the loops in $\widetilde{\mathcal{L}}_{1/2} \cdot \mathbb{1}_{\widetilde{\ell} \cap B(n) \neq \emptyset}$.
- ($\widehat{\mathbf{\Psi}}_n$ as a cut set) On the event $\left\{\mathbf{0} \xleftrightarrow{\le [(1+\lambda)N]^b} \partial B((1+\lambda)N)\right\}$, there exists $x \in \overline{\mathbf{\Psi}}_n^* := \widehat{\mathbf{\Psi}}_n \cap B(n + [(1+\lambda)N]^b) \setminus B(n-1)$ such that $B_x(1)$ is connected to $\partial B((1+\lambda)N)$ by the loops in $\widetilde{\mathcal{L}}_{1/2}$ that are not used to construct $\widehat{\mathbf{\Psi}}_n$.
- Let $\psi_n^* := |\overline{\mathbf{\Psi}}_n^*|$ and $L := \epsilon^{\frac{3}{10}} N$. We define

$$\mathsf{B}_2 := \left\{\exists n \in [(1+\tfrac{\lambda}{4}), (1+\tfrac{\lambda}{3})] \text{ s.t. } 0 < \psi_n^* \le L^2\right\} \cap \left\{\mathbf{0} \xleftrightarrow{\le [(1+\lambda)N]^b} \partial B((1+\lambda)N)\right\}.$$

$$\Rightarrow \mathbb{P}[\mathsf{B}_2] \le C_5 L^2 \mathbb{P}\left[\exists n \in [(1+\tfrac{\lambda}{4}), (1+\tfrac{\lambda}{3})] \text{ s.t. } 0 < \psi_n^* \le L^2\right] \theta(\frac{\lambda N}{2})$$

$$\le C_5 \epsilon^{\frac{3}{5}} N^2 \theta(\tfrac{\lambda N}{2}) \theta(N).$$

Proof of the upper bound 4: regularity theorem

- Let $\chi_n := \left|\{x \in B(n+L) \setminus B(n) : \mathbf{0} \leftrightarrow x\}\right|$. We define

$$\mathbf{B}_3 := \left\{\forall n \in [(1+\tfrac{\lambda}{4})N, (1+\tfrac{\lambda}{3})N], \psi_n^* > L^2\right\} \cap \left\{|\mathbf{C}(\mathbf{0})| < \epsilon N^4\right\}$$

- We need the regularity theorem (core of our proof):

$$\mathbb{P}\left[\psi_n^* \geq L^2, \chi_n \leq c_3 L^4\right] \leq (1-c_4)\theta(N). \tag{10}$$

- For any $i \in \mathbb{N}$, let $n_i := (1+\frac{\lambda}{4})N + iL$. We define

$$I := \left|\{i \in \mathbb{N} : n_i \in [(1+\tfrac{\lambda}{4})N, (1+\tfrac{\lambda}{3})N], \psi_{n_i}^* \geq L^2, \chi_{n_i} \leq c_3 L^4\}\right|.$$

- (10) $\Rightarrow \mathbb{E}(I) \leq \frac{1}{12}\lambda\epsilon^{-\frac{3}{10}}(1-c_4)\theta(N)$.
- On $\left\{|\mathbf{C}(\mathbf{0})| < \epsilon N^4\right\}$, the number of n_i with $\chi_{n_i} > c_3 L^4$ is at most $\frac{\epsilon N^4}{c_3 L^4}$.

$$\Rightarrow \mathbb{P}[\mathbf{B}_3] \leq \mathbb{P}\left[I \geq \tfrac{1}{12}\lambda\epsilon^{-\frac{3}{10}} - \tfrac{\epsilon N^4}{c_3 L^4}\right] \leq \frac{\mathbb{E}(I)}{\frac{1}{12}\lambda\epsilon^{-\frac{3}{10}} - \frac{\epsilon N^4}{c_3 L^4}} \leq (1-c_2)\theta(N).$$

Proof of the upper bound 5: proportion of regular points

- For any $x \in \mathbb{Z}^d$, we say x is a regular point if the sparsity of $\widetilde{\mathbb{Z}}^d \setminus \widehat{\mathbf{\Psi}}_n$ in every box $B_x(R)$ is comparable to $\widetilde{\mathbb{Z}}^d$.
- The proof of the regularity theorem is implemented in two steps:
  1. Prove that with high probability a significant portion of the lattice points in $\overline{\mathbf{\Psi}}_n^*$ are regular.
  2. Employ the second moment method to show that in average each regular $x \in \overline{\mathbf{\Psi}}_n^*$ may contribute $O(L^2)$ lattice points to the cluster $\mathbf{C}(\mathbf{0})$. (**P.S.** $\sum_{y \in B_x(L)} \mathbb{P}[x \leftrightarrow y] \asymp L^2$)
- Step 1 is the most challenging part due to the considerable correlation between the regularity between different x.(key: create independence)
- Our solution:
  - a. multi-scale analysis (k-unqualified points)
  - b. localization for the definition of regular points
  - c. exploration process

Proof of the upper bound 6: conclusion

The event $\{\mathbf{0} \leftrightarrow \partial B((1+\lambda)N)\}$ is decomposed into

$$\mathsf{B}_0 := \left\{\mathbf{0} \leftrightarrow \partial B((1+\lambda)N)\right\} \cap \left\{\mathbf{0} \xleftrightarrow{\le [(1+\lambda)N]^b} \partial B((1+\lambda)N)\right\}^c,$$

$$\mathbf{B}_1 := \left\{|\mathbf{C}(\mathbf{0})| \ge \epsilon N^4\right\},$$

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$$\mathbf{B}_3 := \left\{\forall n \in [(1+\tfrac{\lambda}{4})N, (1+\tfrac{\lambda}{3})N], \psi_n^* > L^2\right\} \cap \left\{|\mathbf{C}(\mathbf{0})| < \epsilon N^4\right\}.$$

Consequently, by combining the estimates for these four sub-events, we complete the induction and obtain the desired upper bound.

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Thanks for your attention!

Questions? Remarks?